

# Photons with half-integral spin as q-Fermions

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## Abstract

The recently discovered 'light (photons) with half-integral spin' is interpreted as q-Fermions proposed by us in 1991, as these q-Fermions satisfy q-deformed anti-commutation relations (pertaining to spin half) and have the property that more than one q-Fermion can occupy a given quantum state. In this article, in view of the recent discovery, we recall the construction of q-Fermions and give the statistical properties of q-Fermion gas, based on our preprint in 1992.

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Recently, Ballantine, Donegan and Eastham [1] reported the discovery of 'light (photons) with half-integral spin'. This new form of light is interpreted as q-Fermions proposed by Parthasarathy and Viswanathan [2] in 1991. The q-Fermions satisfy q-deformed anti-commutation relations pertaining to spin half and have the novel property that more than one q-Fermion can occupy a given state. In the limit  $q \rightarrow 1$ , we recover ordinary fermions. With  $q \neq 1$ , the q-Fermion system could be the right description of the newly discovered light with half-integral spin. We [3] have given the q-deformed statistics of q-Fermions, obtaining the number density, chemical potential and the specific heat  $C_V$ . In view of the recent discovery, we give the statistical properties of q-Fermion gas, essentially taken from the preprint [3].

The result in [1] shows, in two dimensions photons can have a half-integer total angular momentum, comprising of unequal mixture of spin and orbital contributions and demonstrate the half-integer quantization of this total angular momentum. They involve beams of light propagating in a particular direction when the full rotational symmetry is not present. The restricted symmetry leads to a new form of total angular momentum which has a half-integer; that is 'fermionic' spectrum. This half-integer spectrum shows that for light, reduced dimensionality allows for new form of quantization. The half-integer quantization, which they demonstrate through noise measurement, implies fermionic exchange statistics. A description of such photons could be provided by the q-Fermions and q-Fermion statistics. In our preprint [3], we have demonstrated that q-deformed version is not merely a mathematical construct but perhaps effectively takes into account the interactions. These interactions are contact interactions and either the *surroundings* or the system itself generate these interactions. It is in this sense that the restricted symmetry is effectively taken into account by the q-deformation. While the q-deformed Boson description maintains the commutation relations though q-deformed, to describe half-integer total angular momentum of the new photons, q-Fermion description is appropriate. Nevertheless, this q-Fermion describing the new photon cannot be a fundamental particle, but particle with interactions included, as if it is a quasi-particle.

## **q-Fermion Algebra - A brief Review**

A q-deformed non-trivial (in the sense that there is another version of

q-fermion algebra [4] which can be transformed to ordinary fermion algebra and hence trivial) fermion algebra proposed by us [2] is given by

$$\begin{aligned} ff^\dagger + \sqrt{q}f^\dagger f &= q^{-\frac{N}{2}}, \\ [N, f] &= -f, \\ [N, f^\dagger] &= f^\dagger, \\ f^2 \neq 0 \quad ; \quad (f^\dagger)^2 \neq 0, \end{aligned} \quad (1)$$

where  $N$  is the q-Fermion number operator  $\neq f^\dagger f$ . The orthonormal  $n$  q-Fermion state is defined by

$$|n\rangle_q^F = \frac{1}{([n]_q^F!)^{\frac{1}{2}}} (f^\dagger)^n |0\rangle, \quad (2)$$

where the vacuum  $|0\rangle$  is defined as  $f|0\rangle = 0$  and

$$\begin{aligned} [n]_q^F! &= [n]_q^F [n-1]_q^F \cdots [2]_q^F [1]_q^F, \\ [n]_q^F &= q^{-\frac{(n-1)}{2}} \sum_{k=0}^{(n-1)} (-q)^k = \frac{\sqrt{q}}{1+q} \left( q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}} \right). \end{aligned} \quad (3)$$

This algebra describes q-fermions such that any number of q-fermions can occupy a given state for  $0 < q < 1$ . Here  $q$  is taken to be real. When  $q = 1$ , all states other than the vacuum and one particle state collapse, thereby recovering Pauli principle. Subsequently, Viswanathan, Parthasarathy and Jagannathan [5] have constructed coherent states for q-fermions. In it, we used transformed q-fermion operator as  $f = q^{-\frac{N}{4}} F$ ,  $f^\dagger = F^\dagger q^{-\frac{N}{4}}$ , so that we find equivalent algebra

$$\begin{aligned} FF^\dagger + qF^\dagger F &= 1, \\ FF^\dagger - F^\dagger F &= (-q)^N, \\ [N, F] &= -F, \\ [N, F^\dagger] &= F^\dagger, \\ F^2 \neq 0 \quad ; \quad (F^\dagger)^2 \neq 0, \end{aligned} \quad (4)$$

from which we see

$$FF^\dagger = [N+1]^f \quad ; \quad F^\dagger F = [N]^f, \quad (5)$$

where

$$[n]^f = \frac{1 - (-q)^n}{1 + q}. \quad (6)$$

The superscript 'f' is used to indicate that we are dealing with q-fermion. Here  $N$  is the number operator,  $\neq F^\dagger F$ . The Fock space  $|n\rangle$  can be constructed as

$$\begin{aligned} F|n\rangle &= ([n]^f)^{\frac{1}{2}}|n-1\rangle, \\ F^\dagger|n\rangle &= ([n+1]^f)^{\frac{1}{2}}|n+1\rangle. \end{aligned} \quad (7)$$

As before, the vacuum is defined by  $F|0\rangle = 0$ . For  $q \neq 1$ , more than one q-Fermion can occupy a given state and *only when  $q = 1$ , all states other than  $|0\rangle$  and  $|1\rangle$  collapse, recovering Pauli principle. The above q-fermion algebra (either in terms of  $f$  or  $F$ ) is non-trivial as it cannot be transformed to ordinary fermion algebra.* Using (6), we find

$$[n+1]^f + q[n]^f = 1. \quad (8)$$

## Many q-Fermion system - q-Fermion Gas

As a model Hamiltonian for many q-fermion system, we take

$$H = \sum_k (E_k - \mu) N_k, \quad (9)$$

where  $E_k$  is the kinetic energy for q-fermion of momentum  $k$ . We assume that q-fermion operators with different momenta commute. We define the thermal average of  $F^\dagger F$  as

$$\langle F_k^\dagger F_k \rangle = \frac{\text{Tr}(\exp(-\beta H) F_k^\dagger F_k)}{\text{Tr} \exp(-\beta H)}, \quad (10)$$

where  $\beta = \frac{1}{KT}$ ,  $K$  the Boltzmann constant. Using the cyclic property of the trace and (4), it follows

$$\langle F_k^\dagger F_k \rangle = \exp(-\beta(E_k - \mu)) \langle F_k F_k^\dagger \rangle. \quad (11)$$

Using (8), we find

$$\langle [N_k]^f \rangle = \frac{1}{\exp(\beta(E_k - \mu)) + q}, \quad (12)$$

which is the distribution function for  $q$ -fermions. It is noted that when  $T \rightarrow 0$ , for  $E_k > \mu$ ,  $\langle [N_k]^f \rangle \rightarrow 0$ ; for  $E_k < \mu$ ,  $\langle [N_k]^f \rangle \rightarrow \frac{1}{q}$  and for  $E_k = \mu$ ,  $\langle [N_k]^f \rangle \rightarrow \frac{1}{1+q}$ . So upto  $E_k = \mu_0 < \mu$ , the levels are filled and empty when  $E_k > \mu$ . Further from (5) and (11), we see that

$$\frac{[N_k]^f}{[N_k + 1]^f} = e^{-\beta(E_k - \mu)}, \quad (13)$$

From (6), in the limit  $q \rightarrow 1$ ,  $[n] = n$ ,  $[n + 1] = 1 - n$ ,  $n = 0, 1$ , and so (13) gives the familiar Fermi-Dirac distribution when  $q = 1$ .

The distribution function (12) for many  $q$ -Fermion system derived by us in Ref.3 (in 1992) has been the subject of study by others, Narayana Swamy [6] in 2006, Algin and Senay [7] in 2012, Algin, Irk and Topcu [8] in 2015. The expression (12) can be solved for  $N_k$  as

$$N_k = \frac{1}{|\ln q|} \left| \ln \left( \frac{|e^{\eta_k} - 1|}{e^{\eta_k} + q} \right) \right|, \quad (14)$$

where  $\eta_k = \beta(E_k - \mu)$ , agreeing with [6], [7].

The Hamiltonian (9) allows us to evaluate the number density  $\rho$  and the internal energy  $u$  for the  $q$ -Fermion gas, by going from discrete sum to integral. Following Feynman [9], we consider

$$I = \int_0^\infty \frac{g(E)dE}{\exp(\beta(E - \mu)) + q}, \quad (15)$$

where  $g(E) = c\sqrt{E}$  for calculating the number density  $\rho$  and  $g(E) = cE\sqrt{E}$  for calculating the internal energy,  $c$  a constant. Splitting the integral as

$$I = \frac{1}{q} \int_0^\mu g(E) dE - \frac{1}{q} \int_0^\mu \frac{g(E) dE}{1 + q \exp(-\beta(E - \mu))} + \int_\mu^\infty \frac{g(E) dE}{\exp(\beta(E - \mu)) + q}, \quad (16)$$

which can be easily verified, setting  $x = -\beta(E - \mu)$  in the second integral and  $x = \beta(E - \mu)$  in the third integral, we find

$$I = \frac{1}{q} \int_0^\mu g(E) dE - \frac{1}{q\beta} \int_0^{\mu\beta} \frac{g(\mu - \frac{x}{\beta}) dx}{1 + qe^x} + \frac{1}{\beta} \int_0^\infty \frac{g(\mu + \frac{x}{\beta}) dx}{e^x + q}, \quad (17)$$

For low enough temperatures  $g(\mu \pm \frac{x}{\beta}) \simeq g(\mu) \pm \frac{x}{\beta} g'(\mu)$  and  $\int_0^{\mu\beta} \rightarrow \int_0^\infty$ . Then

$$I = \frac{1}{q} \int_0^\mu g(E) dE - \frac{g(\mu)}{q\beta} \int_0^\infty \frac{dx}{1 + qe^x} + \frac{g'(\mu)}{q\beta^2} \int_0^\infty \frac{xdx}{1 + qe^x} + \frac{g(\mu)}{\beta} \int_0^\infty \frac{dx}{e^x + q} + \frac{g'(\mu)}{\beta^2} \int_0^\infty \frac{xdx}{e^x + q}. \quad (18)$$

Now

$$\int_0^\infty \frac{dx}{1 + qe^x} = \ln \left( \frac{1 + q}{q} \right) \quad ; \quad \int_0^\infty \frac{xdx}{e^x + q} = \frac{1}{q} \ln(1 + q). \quad (19)$$

Collecting  $g'(\mu)$  terms, we have

$$\frac{g'(\mu)}{\beta^2} \left( \frac{1}{q} \int_0^\infty \frac{xdx}{1 + qe^x} + \int_0^\infty \frac{xdx}{e^x + q} \right), \quad (20)$$

which can be evaluated as

$$\frac{g'(\mu)}{q\beta^2} \left( \frac{\pi^2}{6} + \frac{1}{2}(\ln q)^2 \right). \quad (21)$$

Thus,

$$I = \frac{1}{q} \int_0^\mu g(E) dE + \frac{g(\mu)}{q\beta} \ln q + \frac{g'(\mu)}{q\beta^2} \left( \frac{\pi^2}{6} + \frac{1}{2}(\ln q)^2 \right). \quad (22)$$

For finding the number density  $\rho$ , set  $g(E) = c\sqrt{E}$  and then we find

$$\rho = \frac{2c}{3q}\mu^{\frac{3}{2}} + \frac{c\sqrt{\mu}}{q\beta}\ell n q + \frac{c}{2q\beta^2\sqrt{\mu}}\left(\frac{\pi^2}{6} + \frac{1}{2}(\ell n q)^2\right). \quad (23)$$

From this, we see

$$\rho_{T=0} = \frac{2c}{3q}\mu_0^{\frac{3}{2}}. \quad (24)$$

Equating  $\rho_{T=0} = \rho_{T \neq 0}$  as required by number conservation, we find

$$\mu^{\frac{3}{2}} = \mu_0^{\frac{3}{2}} \left(1 + \frac{3}{2\mu\beta}\ell n q + \frac{3}{4\beta^2\mu^2}\left[\frac{\pi^2}{6} + \frac{1}{2}(\ell n q)^2\right]\right)^{-1}. \quad (25)$$

Here we approximate  $\mu$  by  $\mu_0$  in the (.....)<sup>-1</sup> and then

$$\mu \simeq \mu_0 \left(1 - \frac{1}{\mu_0\beta}\ell n q - \frac{\pi^2}{12\mu_0^2\beta^2} + \frac{1}{\mu_0^2\beta^2}(\ell n q)^2\right). \quad (26)$$

The internal energy of q-fermion gas is evaluated by taking  $g(E) = cE\sqrt{E}$  and we find

$$u = \frac{2c}{5q}\mu^{\frac{5}{2}} + \frac{c}{q\beta}\mu^{\frac{3}{2}}\ell n q + \frac{3c}{2q\beta^2}\sqrt{\mu}\left[\frac{\pi^2}{6} + \frac{1}{2}(\ell n q)^2\right]. \quad (27)$$

Using the expression for  $\mu$  above and after some steps, the internal energy becomes

$$u = u_0 + \gamma T^2, \quad (28)$$

where

$$\begin{aligned} u_0 &= \frac{2c}{5q}\mu_0^{\frac{5}{2}}, \\ \gamma &= \frac{c\sqrt{\mu_0}K^2}{q} \frac{\pi^2}{6} + \frac{c\sqrt{\mu_0}K^2}{q}(\ell n q)^2, \\ &= \frac{c\sqrt{\mu_0}K^2\pi^2}{6} \left(\frac{1}{q} + \frac{6}{\pi^2 q}(\ell n q)^2\right). \end{aligned} \quad (29)$$

The pre-factor in  $\gamma$  is the Feynman's value and the correction is multiplicative parenthesis. Since  $C_V = \frac{\partial u}{\partial T}$ , we find for q-fermion gas

$$C_V = C_V^{Feynman} \left( \frac{1}{q} + \frac{6}{\pi^2 q} (\ln q)^2 \right). \quad (30)$$

In the limit  $q = 1$ , we recover Feynman's value. The results in (23), (25), (26), (27) and (30) were derived by us in our preprint [3].

By measuring  $C_V$  for q-fermion gas,  $q$  can be determined which can then be used to find  $\rho$ . These expressions can be applied to the 'newly discovered photon with half-integral spin', if the  $\rho$ ,  $C_V$  are measured for this system. Further, the expression for  $N_k$  in (14) can be used to plot the q-deformed statistical distribution function for q-Fermion gas for various  $T$  as a function of  $\beta(E - \mu)$  for values of  $q < 1$ . This plot is available from Fig.1 of [7] and can be used for the distribution function of the 'new light with half-integer spin'.

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